

# Chapter 16

## Manifolds and Geodesics

Reading:

- Osserman [7] Pg. 43-52, 55, 63-65,
- Do Carmo [2] Pg. 238-247, 325-335.

### 16.1 Manifold Theory

Let us recall the definition of differentiable manifolds

**Definition 16.1.1.** *An **n-manifold** is a Hausdorff space, each point of which has a neighborhood homeomorphic to a domain in  $\mathbb{R}^n$ .*

**Definition 16.1.2.** *An **atlas**  $A$  for an  $n$ -manifold  $M^n$  is a collection of triples  $(U_\alpha, V_\alpha, \varphi_\alpha)$  where  $U_\alpha$  is a domain in  $\mathbf{R}^n$ ,  $V_\alpha$  is an open set on  $M^n$ , and  $\varphi_\alpha$  is a homeomorphism of  $U_\alpha$  onto  $V_\alpha$ , and*

$$\bigcup_{\alpha} V_{\alpha} = M^n \quad (16.1)$$

*Each triple is called a **map**.*

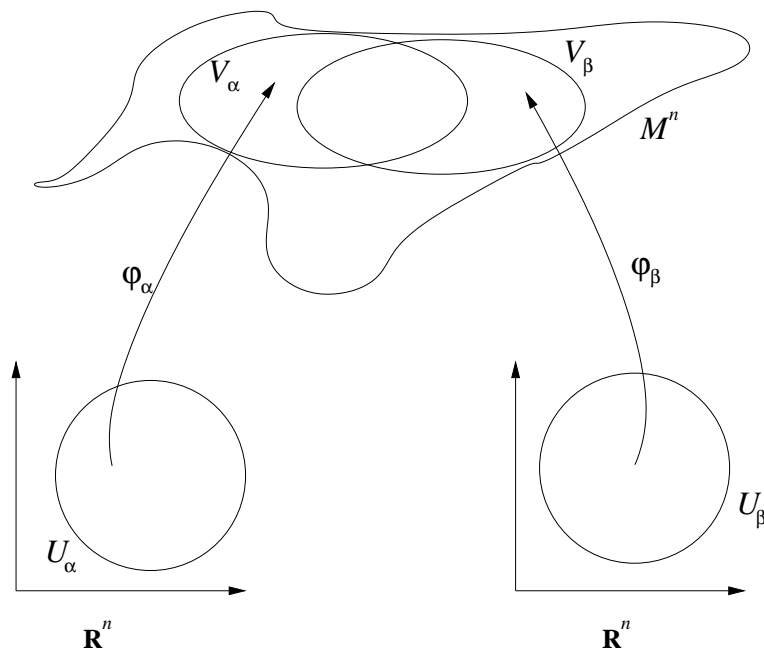


Figure 16.1: Definition of atlas

**Definition 16.1.3.** A  $C^r$ - (resp. conformal) structure on  $M^n$  is an atlas for which each transformation  $\varphi_\alpha^{-1} \circ \varphi_\beta \in C^r$  (resp. conformal) wherever it is defined.

**Corollary 16.1.4.** The space  $\mathbf{R}^n$  has a canonical  $C^r$ -structure for all  $r$ , defined by letting  $A$  consists of the single triple  $U_\alpha = V_\alpha = \mathbf{R}^n$ , and  $\varphi_\alpha$  the identity map.

Let  $S$  be a  $C^r$ -surface in  $\mathbf{R}^n$ , and  $A$  the  $C^r$ -structure on the associated 2-manifold  $M$ . We discussed that all local properties of surfaces which are independent of parameters are well defined on a global surface  $S$  by the change of parameters. The global properties of  $S$  will be defined simply to be those of  $M$ , such as orientation, compactness, connectedness, simply connectedness, etc.

In the rest of the course, all surfaces will be connected and orientable.

**Definition 16.1.5.** A regular  $C^2$ -surface  $S$  in  $\mathbf{R}^n$  is a **minimal surface** if its mean curvature vector vanishes at each point.

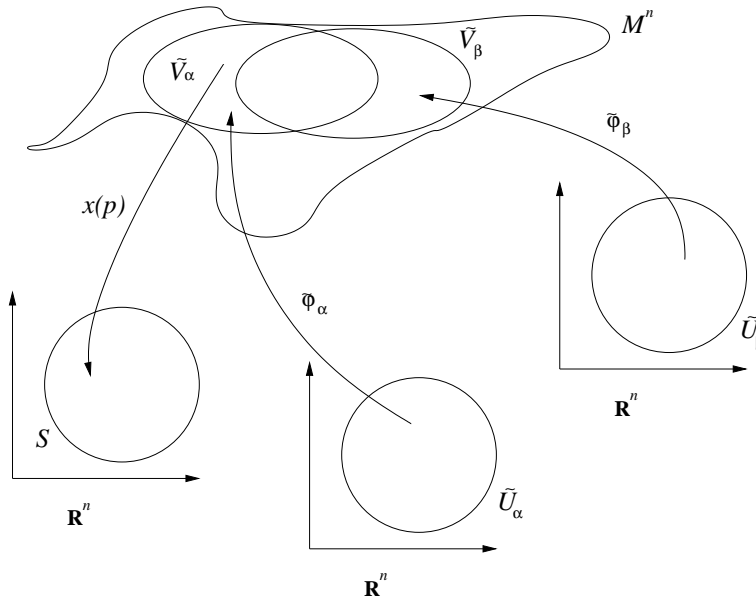


Figure 16.2: Lemma 6.1

The following two lemmas are useful for the proof of Lemma 6.1 in [7].

**(Lemma 4.4 in [7]).** Let  $S$  be a minimal surface. Every regular point of  $S$  has a neighborhood in which there exists a reparametrization of  $S$  in terms of isothermal parameters.

**(Lemma 4.5 in [7]).** Let a surface  $S$  be defined by  $x(u)$ , where  $u_1, u_2$  are isothermal parameters, and let  $\tilde{S}$  be a reparametrization of  $S$  defined by a diffeomorphism  $u(\tilde{u})$ . Then  $\tilde{u}_1, \tilde{u}_2$  are also isothermal parameters if and only if the map  $u(\tilde{u})$  is either conformal or anti-conformal.

**(Lemma 6.1 in [7]).** Let  $S$  be a regular minimal surface in  $\mathbf{R}^n$  defined by a map  $x(p) : M \rightarrow \mathbf{R}^n$ . Then  $S$  induces a conformal structure on  $M$ .

*Proof.* Assume the surface  $S$  is orientable, and  $A$  be an oriented atlas of  $M$ . Let  $\tilde{A}$  be the collection of all the maps  $(\tilde{U}_\alpha, \tilde{V}_\alpha, \tilde{\varphi}_\alpha) \in A$  such that  $\tilde{\varphi}_\beta^{-1} \circ \tilde{\varphi}_\alpha$  preserves orientation wherever defined, and the map  $x \circ \tilde{\varphi}_\alpha : \tilde{U}_\alpha \rightarrow \mathbf{R}^n$  defines a local surface in isothermal parameters. By Lemma 16.1 the union of  $\tilde{V}_\alpha$  equals  $M$ , so  $\tilde{A}$  is an atlas for  $M$ . And by Lemma 16.1 each  $\tilde{\varphi}_\beta \circ \tilde{\varphi}_\alpha$  is conformal wherever defined. So  $\tilde{A}$  defines a conformal structure on  $M$ .  $\square$

With the previous lemma, we can discuss some basic notions connected with conformal structure. If  $M$  has a conformal structure, then we can define all concepts which are invariant under conformal mapping, such as analytic maps of one such manifold  $M$  into another  $\tilde{M}$ .

**Example 7.** (*Stereographic Projection*) A **meromorphic function** on  $M$  is a complex analytic map of  $M$  into the Riemann sphere. The latter can be defined as the unit sphere in  $\mathbf{R}^3$  with the conformal structure defined by a pair of maps

$$\varphi_1 : x = \left( \frac{2u_1}{|w|^2 + 1}, \frac{2u_2}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right), \quad w = u_1 + iu_2 \quad (16.2)$$

$$\varphi_2 : x = \left( \frac{2\tilde{u}_1}{|\tilde{w}|^2 + 1}, \frac{-2\tilde{u}_2}{|\tilde{w}|^2 + 1}, \frac{1 - |\tilde{w}|^2}{|\tilde{w}|^2 + 1} \right), \quad \tilde{w} = \tilde{u}_1 + i\tilde{u}_2 \quad (16.3)$$

The map  $\varphi_1$  is called the **stereographic projection** from the point  $(0, 0, 1)$ , and one can easily show that  $\varphi_1^{-1} \circ \varphi_2$  is simply  $w = \frac{1}{\tilde{w}}$ , a conformal map of  $0 < |\tilde{w}| < \infty$  onto  $0 < |w| < \infty$ .

**Definition 16.1.6.** A **generalized minimal surface**  $S$  in  $\mathbf{R}^n$  is a non-constant map  $x(p) : M \rightarrow \mathbf{R}^n$ , where  $M$  is a 2-manifold with a conformal structure defined by an atlas  $A = \{U_\alpha, V_\alpha, \varphi_\alpha\}$ , such that each coordinate function  $x_k(p)$  is harmonic on  $M$ , and furthermore

$$\sum_{k=1}^n \phi_k^2(\zeta) = 0 \quad (16.4)$$

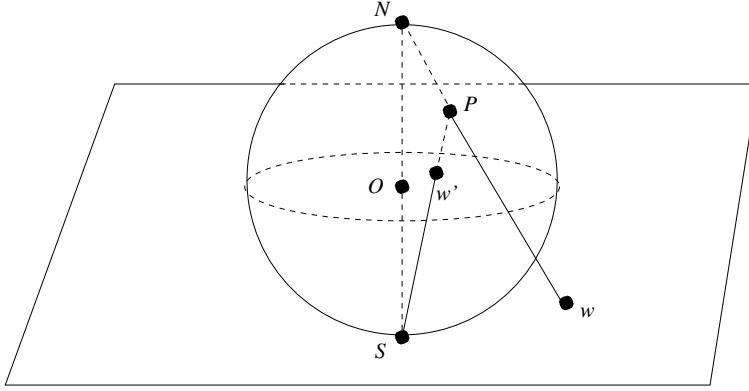


Figure 16.3: Stereographic Projection

where we set for an arbitrary  $a$ ,

$$h_k(\zeta) = x_k(\varphi_\alpha(\zeta)) \quad (16.5)$$

$$\phi_k(\zeta) = \frac{\partial h_k}{\partial \xi_1} - i \frac{\partial h_k}{\partial \xi_2}, \quad \zeta = \xi_1 + i\xi_2 \quad (16.6)$$

Following is a lemma from Ch.4 in [7]

**(Lemma 4.3 in [7]).** *Let  $x(u)$  define a regular minimal surface, with  $u_1, u_2$  isothermal parameters. Then the function  $\phi_k(\zeta)$  defined by 16.6 are analytic, and they satisfy equation*

$$\sum_{k=1}^n \phi_k^2(\zeta) = 0 \quad (16.7)$$

and

$$\sum_{k=1}^n |\phi_k^2(\zeta)| \neq 0. \quad (16.8)$$

*Conversely, let  $\phi_1(\zeta), \dots, \phi_n(\zeta)$  be analytic functions of  $\zeta$  which satisfy Eqs. 16.7 and 16.8 in a simply-connected domain  $D$ . Then there exists a regular minimal surface  $x(u)$  defined over  $D$ , such that Eqs. 16.6 are valid.*

**Corollary 16.1.7.** *If  $S$  is regular minimal surface, then  $S$  is also a generalized minimal surface.*

*Proof.* We can use the conformal structure defined in Lemma 6.1, and the result follows from Lemma 4.3 in [7]  $\square$

**Definition 16.1.8.** *Let  $S$  be a generalized minimal surface, and  $\zeta \in S$ . The **branch points**  $\zeta$ 's with respect to the function  $\phi_k$  correspond to the  $\zeta$ 's at which*

$$\sum_{k=1}^n |\phi_k^2(\zeta)| = 0 \quad (16.9)$$

**Corollary 16.1.9.** *Let  $S$  be a generalized minimal surface, and  $S'$  be the surface  $S$  with branch points with respect to the function  $\phi$  in Eq. 16.6 deleted. Then  $S'$  is a regular minimal surface.*

*Proof.* Let  $x(p)$  be the coordinate map of  $S$ , where  $p \in S$ . Since  $x(p)$  is non constant, at least one of the function  $x_k(p)$  is non constant. That means that the corresponding  $\phi_k(\zeta)$  can have at most isolated zeroes, and the equation

$$\sum_{k=1}^n |\phi_k^2(\zeta)| = 0 \quad (16.10)$$

can hold at most at the branch points. Since  $S'$  consists of the whole surface  $S$  without the branch points,  $S'$  is a regular minimal surface, from Lemma 4.3 in [7].  $\square$

In the case of  $n = 2$  in the definition of a generalized surface, either  $x_1 + ix_2$  or  $x_1 - x_2$  is a non-constant analytic function  $f(\zeta)$ . The branch points on the surface satisfy the Eq. 16.9. That is, they satisfy the equation  $f'(\zeta) = 0$ , which is the inverse mapping.

For large  $n$ , the difference between regular and generalized minimal surfaces consists in allowing the possibility of isolated branch points. However, there are theorems where the possible existence of branch points has no effect. The following lemma is one of the example.

**(Lemma 6.2 in [7]).** *A generalized minimal surface cannot be compact*

*Proof.* Let  $S$  be a generalized minimal surface defined by a map  $x(p) : M \rightarrow \mathbf{R}^n$ . Then each coordinate function  $x_k(p)$  is harmonic on  $M$ . If  $M$  were compact, the function  $x_k(p)$  would attain its maximum, hence it would be a constant. This contradicts the assumption that the map  $x(p)$  is non-constant.  $\square$

## 16.2 Plateau Problem

One of the prime examples of extending the properties of generalized surface to regular surface is the classical Plateau problem, which is discussed in the appendix of [7].

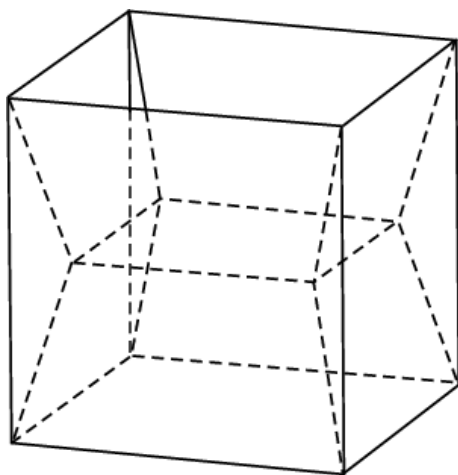


Figure 16.4: A 13-polygon surface obtained for a cubical wire frame

**Definition 16.2.1.** An arc  $z(t)$  is **simple** if  $z(t_1) = z(t_2)$  only for  $t_1 = t_2$ . A **Jordan curve** is a simple closed curve.

**Proposition 16.2.2.** (Osserman) Let  $\Gamma$  be an arbitrary Jordan curve in  $\mathbf{R}^3$ . Then there exists a regular simply connected minimal surface bounded by  $\Gamma$ .

The existence of a solution to the general case was independently proven by Douglas (1931) [3] and Radò (1933) [8], although their analysis could

not exclude the possibility of singularities (i.e. for the case of generalized minimal surface). Osserman (1970) [6] and Gulliver (1973) [4] showed that a minimizing solution cannot have singularities [9].

Table 16.1: Development on the Plateau's problem in 1970-1985	
Meeks and Yau	When the curve $\Gamma$ defined in Prop. 16.2.2 lies on the boundary of a convex body, then the surface obtained is embedded (i.e. without self-intersections).
Gulliver and Spruck	They proved the result from Meeds and Yau under the additional assumption that the total curvature of $\Gamma$ was at most $4\pi$ .

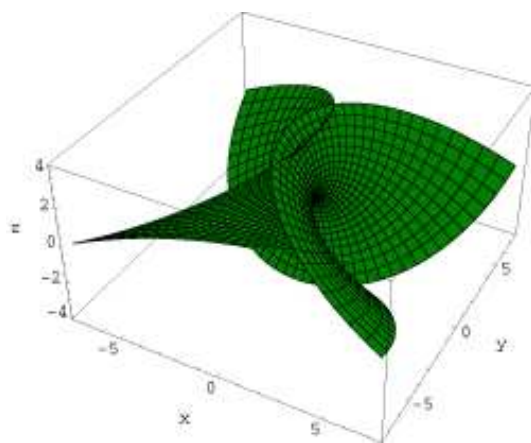


Figure 16.5: An Enneper surface

## 16.3 Geodesics

A geodesics is analogue to the straight line on a Euclidean plane. In order to define geodesics, we first have to understand the notion of covariant derivative, which is analogue to the usual differentiation of vectors in the plane.



**Definition 16.3.1.** A vector field  $w$  in an open set  $U$  in the regular surface  $S$  assigns to each  $p \in U$  a vector  $w(p) \in T_p(S)$ . The vector field is **differentiable** at  $p$  if, for some parametrization  $\mathbf{x}(u, v)$  in  $p$ , the components  $a$  and  $b$  of  $w = a\mathbf{x}_u + b\mathbf{x}_v$  in the basis of  $\{\mathbf{x}_u, \mathbf{x}_v\}$  are differentiable functions at  $p$ . The vector field is differentiable in  $U$  if it is differentiable for all  $p \in U$ .

**Definition 16.3.2.** Let  $w$  be a differentiable vector field in an open set  $U \subset S$  and  $p \in U$ . Let  $y \in T_p(S)$  and  $\alpha : (-\epsilon, \epsilon) \rightarrow U$  a parametrized curve with  $\alpha(0) = p$  and  $\alpha'(0) = y$ . Let  $w(t)$  be the restriction of the vector field  $w$  to the curve  $\alpha$ . Then the **covariant derivative** at  $p$  of the vector field  $w$  relative to the vector  $y$ ,  $(Dw/dt)(0)$ , is given by the vector obtained by the normal projection of  $(dw/dt)(0)$  onto the plane  $T_p(S)$ .

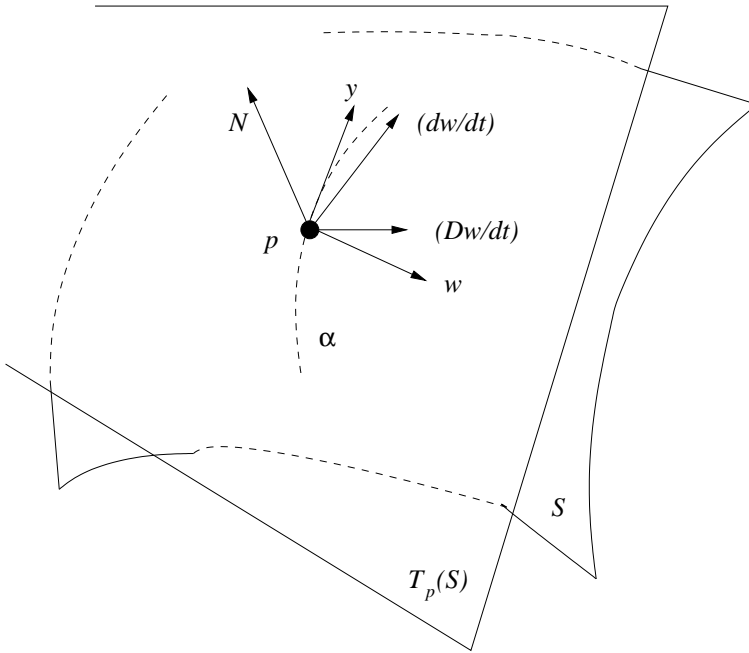


Figure 16.6: The covariant derivative

**Definition 16.3.3.** A vector field  $w$  along a parametrized curve  $\alpha : I \rightarrow S$  is said to be **parallel** if  $Dw/dt = 0$  for every  $t \in I$ .

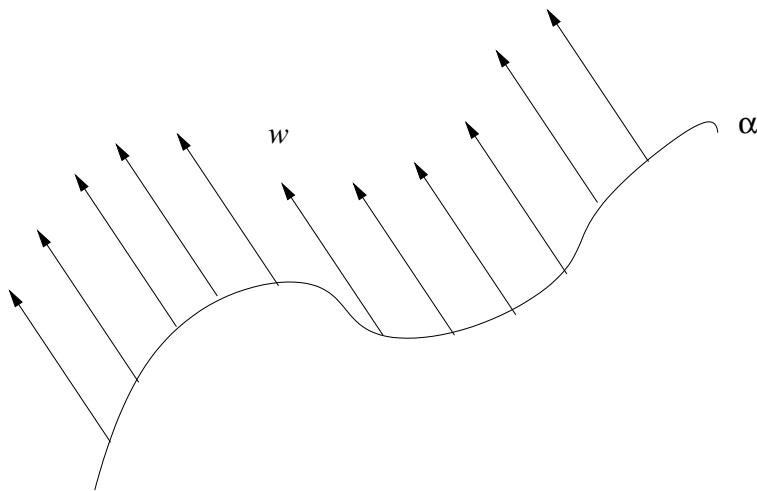


Figure 16.7: A parallel vector field  $w$  along the curve  $\alpha$ .

**Definition 16.3.4.** A non-constant, parametrized curve  $\gamma : I \rightarrow S$  is said to be **geodesic** at  $t \in I$  if the field of its tangent vectors  $\gamma'(t)$  is parallel along  $\gamma$  at  $t$ , that is

$$\frac{D\gamma'(t)}{dt} = 0 \quad (16.11)$$

From Eq. 16.11, we know that  $|\gamma'(t)| = \text{constant}$ , thus the arc length  $s$  is proportional to the parameter  $t$ , and thus we can reparametrize  $\gamma$  with parameter  $s$ . Note also that Eq. 16.11 implies that  $\alpha''(s) = kn$  is normal to the tangent plane, or parallel to the normal to the surface. Therefore another way to define a geodesic is a regular curve which its principal normal at each point  $p$  along the curve is parallel to the normal to  $S$  at  $p$ .

Below are some examples of geodesics:

**Example 8 (Geodesics on the sphere  $S^2$ ).** The great circles  $C$  of a sphere  $S^2$  are obtained by intersecting the sphere with a plane that passes through the center  $O$  of the sphere. The principal normal at a point  $p \in C$  lies in the direction of the line that connects  $p$  to  $O$ , the center of  $C$ . Since this is also the direction of the normal at  $p$ , the great circles are geodesics.

**Example 9 (Geodesics on a right circular cylinder over the circle  $x^2 + y^2 = 1$ ).** *It is clear that the circles obtained by the intersection of the cylinder with planes that are normal to the axis of the cylinder are geodesics. The straight lines of the cylinder are also geodesics. To find other geodesics on the cylinder  $C$ , consider the parametrization*

$$\mathbf{x}(u, v) = (\cos u, \sin u, v) \quad (16.12)$$

*of the cylinder in a point  $p \in C$ , with  $\mathbf{x}(0, 0) = p$ . Then  $\mathbf{x}$  is an isometry that maps a neighborhood  $U$  of  $(0, 0)$  of the  $uv$ -plane into the cylinder. Since the condition of being a geodesic is local and invariant by isometries, the image of straight lines in  $U$  under the map  $\mathbf{x}$  should be a geodesic on  $C$ . Since a straight line on the  $uv$ -plane can be expressed as*

$$u(s) = as, \quad v(s) = bs, \quad a^2 + b^2 = 1, \quad (16.13)$$

*it follows that a geodesic of the cylinder is locally of the form*

$$(\cos as, \sin as, bs) \quad (16.14)$$

*which is a helix.*

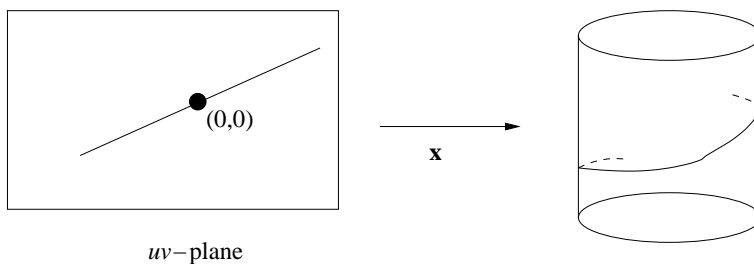


Figure 16.8: Geodesics on a cylinder

## 16.4 Complete Surfaces

In order to study regular surfaces globally, we need some global hypothesis to ensure that the surface cannot be extended further as a regular surface. Compactness serves this purpose, but it would be useful to have a weaker hypothesis than compactness which could still have the same effect.

**Definition 16.4.1.** *A regular (connected) surface  $S$  is said to be **extendable** if there exists a regular (connected) surface  $\bar{S}$  such that  $S \subset \bar{S}$  as a proper subset. If there exists no such  $\bar{S}$ , then  $S$  is said to be **nonextendable**.*

**Definition 16.4.2.** *A regular surface  $S$  is said to be **complete** when for every point  $p \in S$ , any parametrized geodesic  $\gamma : [0, \epsilon) \rightarrow S$  of  $S$ , starting from  $p = \gamma(0)$ , may be extended into a parametrized geodesic  $\bar{\gamma} : \mathbf{R} \rightarrow S$ , defined on the entire line  $\mathbf{R}$ .*

**Example 10 (Examples of complete/non-complete surface).** 1. *The plane is a complete surface.*

2. *The cone minus the vertex is a noncomplete surface, since by extending a generator (which is a geodesic) sufficiently we reach the vertex, which does not belong to the surface.*

3. *A sphere is a complete surface, since its parametrized geodesics (the great circles) may be defined for every real value.*

4. *The cylinder is a complete surface since its geodesics (circles, lines and helices) can be defined for all real values*

5. *A surface  $S - \{p\}$  obtained by removing a point  $p$  from a complete surface  $S$  is not complete, since there exists a geodesic of  $S - \{p\}$  that starts from a point in the neighborhood of  $p$  and cannot be extended through  $p$ .*

**Proposition 16.4.3.** *A complete surface  $S$  is nonextendable.*

*Proof.* Let us assume that  $S$  is extendable and obtain a contradiction. If  $S$  is extendable, then there exists a regular (connected) surface  $\bar{S}$  such that  $S \subset \bar{S}$ . Since  $S$  is a regular surface,  $S$  is open in  $\bar{S}$ . The boundary  $\text{Bd}(S)$  of  $S$  is nonempty, so there exists a point  $p \in \text{Bd}(S)$  such that  $p \notin S$ .

Let  $\bar{V} \subset \bar{S}$  be a neighborhood of  $p$  in  $\bar{S}$  such that every  $q \in \bar{V}$  may be joined to  $p$  by a unique geodesic of  $\bar{S}$ . Since  $p \in \text{Bd}(S)$ , some  $q_0 \in \bar{V}$  belongs to  $S$ . Let  $\bar{\gamma} : [0, 1] \rightarrow \bar{S}$  be a geodesic of  $\bar{S}$ , with  $\bar{\gamma}(0) = p$  and  $\bar{\gamma}(1) = q_0$ . It is clear that  $\alpha : [0, \epsilon) \rightarrow S$ , given by  $\alpha(t) = \bar{\gamma}(1 - t)$ , is a geodesic of  $S$ , with  $\alpha(0) = q_0$ , the extension of which to the line  $\mathbf{R}$  would pass through  $p$  for  $t = 1$ . Since  $p \notin S$ , this geodesic cannot be extended, which contradicts the hypothesis of completeness and concludes the proof.  $\square$

**Proposition 16.4.4.** *A closed surface  $S \subset \mathbf{R}^3$  is complete*

**Corollary 16.4.5.** *A compact surface is complete.*

**Theorem 16.4.6 (Hopf-Rinow).** *Let  $S$  be a complete surface. Given two points  $p, q \in S$ , there exists a minimal geodesic joining  $p$  to  $q$ .*

## 16.5 Riemannian Manifolds

**Definition 16.5.1.** *A Riemannian structure on  $M$ , or a  $C^q$ -Riemannian metric is a collection of matrices  $G_\alpha$ , where the elements of the matrix  $G_\alpha$  are  $C^q$ -functions on  $V_\alpha$ ,  $0 \leq q \leq r - 1$ , and at each point the matrix  $G_\alpha$  is positive definite, while for any  $\alpha, \beta$  such that the map  $u(\tilde{u}) = \varphi_\alpha^{-1} \circ \varphi_\beta$  is defined, the relation*

$$G_\beta = U^T G_\alpha U \quad (16.15)$$

*must hold, where  $U$  is the Jacobian matrix of the transformation  $\varphi_\alpha^{-1} \circ \varphi_\beta$ .*